Thus, in the sense of the foregoing paragraph, we can refer to $\partial x^\mu/\partial \bar{x}^0$ as v^μ and to $(\partial x^\nu/\partial \bar{x}^0)R_{\mu\nu}$ as w_μ. Then (15.176) can be written as

$$-\bar{R}_{00} = v^\mu w_\mu \tag{15.178}$$

where v^μ and w^μ are timelike, spacelike, or null together. It is clear from (15.178) that the covariant requirement that $R_{\alpha\beta}$ carry v_α into a vector w_α such that $v^\alpha w_\alpha \geq 0$ is equivalent to the requirement that $\bar{R}_{00}$ be negative-definite. This can be interpreted physically by saying that $-R_{\alpha\beta}$ must carry the *forward* light cone into itself and the *backward* light cone into itself. The negativeness of the component R_{00} is thereby guaranteed in a covariant way for any real coordinate system:

$$R_{00} \leq 0 \qquad \text{(real coordinates)} \tag{15.179}$$

One must be careful to include the above restriction to a real coordinate system. For instance, if one goes from real x^0 to imaginary $\bar{x}^0$, then $\frac{\partial x^0}{\partial \bar{x}^0}\frac{\partial x^0}{\partial \bar{x}^0}$ is clearly negative, and the statement cannot be true.

The conditions (15.163), (15.165), and (15.179) were first obtained by Rainich in 1925, using somewhat different matrix methods than we have used. They are usually referred to as the *algebraic Rainich conditions.*

The Rainich conditions are purely algebraic, in the sense that they follow entirely from the form of the matrix $T_{\mu\nu}$ as constructed from the antisymmetric matrix $F_{\mu\nu}$ at a single world-point P. They do not involve the change in any quantities as one moves from world-point to world-point. We next need to take into account the fact that $F_{\mu\nu}$ obeys the Maxwell equations, which we now write in the form

$$F^{\mu\nu}{}_{\|\nu} = 0 \qquad {}^*F^{\mu\nu}{}_{\|\nu} = 0 \tag{15.180}$$

At P the coordinates are locally geodesic, and therefore the Christoffel symbols vanish. This allows us to drop the distinction between covariant and ordinary derivatives of first order. Furthermore, since the metric tensor is the Kronecker $\delta_{\mu\nu}$, it is possible to ignore index position. Maxwell's equations at P then can be written as

$$F_{\mu\nu|\nu} = 0 \qquad {}^*F_{\mu\nu|\nu} = 0 \tag{15.181}$$

where we have used the Einstein summation convention without regard for index position.

Our aim now is to study how Eqs. (15.181) reflect themselves in properties of the $T_{\mu\nu}$ and $R_{\mu\nu}$ tensors. Observe that, given the energy-

momentum tensor $T_{\mu\nu}$, we can obtain the generating tensor $F_{\mu\nu}$ only up to a parameter α according to (15.156). At every point x^ν the value of $\lambda_1^2 - \lambda_2^2$ is determined by the energy-momentum tensor $T_{\mu\nu}$, while α may be chosen as a function of x^ν. Algebraically, this α field could be completely incoherent at different world-points. However, since $F_{\mu\nu}$ must satisfy Eq. (15.181), the α becomes a determined point function for which a differential system in terms of $T_{\mu\nu}$ or $R_{\mu\nu}$ can be given. The integrability condition on this system leads to a set of differential conditions on the tensor R, which was first discovered by Misner and Wheeler in 1957.

In order to derive the Misner-Wheeler equations we observe that, if the electromagnetic tensor is written

$$F_{\mu\nu} = (r\cosh\alpha)p_{\mu\nu} + (r\sinh\alpha)q_{\mu\nu} \tag{15.182}$$

as in (15.156), where $r = \sqrt{\lambda_1^2 - \lambda_2^2}$, then its dual is

$${}^*F_{\mu\nu} = (r\cosh\alpha)q_{\mu\nu} + (r\sinh\alpha)p_{\mu\nu} \tag{15.183}$$

This follows from considering the effect of the * operation on the simple forms

$$\bar{p}_{\mu\nu} = \begin{pmatrix} iJ & 0 \\ 0 & 0 \end{pmatrix} \qquad \bar{q}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & iJ \end{pmatrix} \tag{15.184}$$

where the interchange of $\bar{p}$ and $\bar{q}$ under the * operation is evident. But this operation is coordinate-invariant and must hold also in any system where p and q are gotten from $\bar{p}$ and $\bar{q}$ by an orthogonal transformation Q. The general result (15.183) is thereby assured.

The choice of our special coordinate system is arbitrary up to an orthogonal transformation. This allows us to assume without loss of generality that at P the matrices p and q are precisely $\bar{p}$ and $\bar{q}$ in (15.184); that is, $Q = I$. It follows, then, from (15.143) that the matrix

$$\Gamma_{\mu\nu} = c^2 T_{\mu\nu}$$

at P is

$$\bar{\Gamma}_{\mu\nu} = \tfrac{1}{2}r^2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad r^2 = \lambda_1^2 - \lambda_2^2 \tag{15.185}$$

The Maxwell equations (15.181) are, then, from (15.182) and (15.183), (15.186a) $$F_{\mu\nu|\nu} = \tilde{p}_{\mu\nu}(r \cosh \alpha)_{|\nu} + \tilde{q}_{\mu\nu}(r \sinh \alpha)_{|\nu} + (r \cosh \alpha)p_{\mu\nu|\nu} + (r \sinh \alpha)q_{\mu\nu|\nu} = 0$$

(15.186b) $$^*F_{\mu\nu|\nu} = \tilde{q}_{\mu\nu}(r \cosh \alpha)_{|\nu} + \tilde{p}_{\mu\nu}(r \sinh \alpha)_{|\nu} + (r \cosh \alpha)q_{\mu\nu|\nu} + (r \sinh \alpha)p_{\mu\nu|\nu} = 0$$

(Clearly, if we differentiate the $p_{\mu\nu}$ and $q_{\mu\nu}$, we must drop the tilde.) To simplify this form of Maxwell's equations, we define at P local vectors Π_μ and K_μ and scalars A and B by

(15.187) $$\Pi_\mu = p_{\mu\nu|\nu} \qquad K_\mu = q_{\mu\nu|\nu} \qquad A = r \cosh \alpha \qquad B = r \sinh \alpha$$

Then (15.186) reads

(15.188a) $$\tilde{p}_{\mu\nu}A_{|\nu} + \tilde{q}_{\mu\nu}B_{|\nu} + A\Pi_\mu + BK_\mu = 0$$

(15.188b) $$\tilde{q}_{\mu\nu}A_{|\nu} + \tilde{p}_{\mu\nu}B_{|\nu} + AK_\mu + B\Pi_\mu = 0$$

Now we multiply (15.188a) by the matrix $\tilde{p}$ and (15.188b) by $\tilde{q}$, add the two, and use (15.152) to obtain

(15.189) $$A_{|\mu} + \tilde{p}_{\mu\nu}\Pi_\nu A + \tilde{p}_{\mu\nu}K_\nu B + \tilde{q}_{\mu\nu}K_\nu A + \tilde{q}_{\mu\nu}\Pi_\nu B = 0$$

Whence, by definition of A and B,

(15.190) $$A[\tilde{p}_{\mu\nu}\Pi_\nu + \tilde{q}_{\mu\nu}K_\nu + (\log r)_{|\mu}] + B[\tilde{q}_{\mu\nu}\Pi_\nu + \tilde{p}_{\mu\nu}K_\nu + \alpha_{|\mu}] = 0$$

Similarly, multiplying (15.188a) by $\tilde{q}$ and (15.188b) by $\tilde{p}$ and adding, we obtain

(15.191) $$A[\tilde{q}_{\mu\nu}\Pi_\nu + \tilde{p}_{\mu\nu}K_\nu + \alpha_{|\mu}] + B[\tilde{p}_{\mu\nu}\Pi_\nu + \tilde{q}_{\mu\nu}K_\nu + (\log r)_{|\mu}] = 0$$

Equations (15.190) and (15.191) have the form $Ax + By = 0$ and $Bx + Ay = 0$. These equations have a nonzero solution for x and y only if the determinant of the coefficients, $A^2 - B^2$, is zero. Since, however, $A^2 - B^2 = \lambda_1^2 - \lambda_2^2 \neq 0$, the only solution is $x = y = 0$, so the coefficients of A and B in (15.190) and (15.191) are both zero. Thus we obtain the following differential conditions on α and r:

(15.192a) $$\alpha_{|\mu} = -(\tilde{q}_{\mu\nu}\Pi_\nu + \tilde{p}_{\mu\nu}K_\nu)$$

(15.192b) $$(\log r)_{|\mu} = -(\tilde{p}_{\mu\nu}\Pi_\nu + \tilde{q}_{\mu\nu}K_\nu)$$

The next step in our development is to calculate the vectors Π_μ and K_μ and to use the results to put (15.192a) into more interesting form. In order to do this we first have to find the derivatives of the matrices p and q. At the world-point P, the matrices p and q are precisely $\tilde{p}$ and $\tilde{q}$. If we consider, however, the world-point $P(\epsilon,\mu)$, obtained by moving a small amount ϵ in the direction of the μth coordinate axis, we shall have matrices $p^{(\mu)}$ and $q^{(\mu)}$, which differ from $\tilde{p}$ and $\tilde{q}$ by a small rotation in space-time corresponding to an orthogonal matrix $Q_{(\mu)}$. Let us represent $Q_{(\mu)}$ by a series

(15.193) $$Q_{(\mu)} = I + \epsilon C_{(\mu)} + \epsilon^2 D_{(\mu)} + \cdots$$

and use the orthogonality condition on $Q_{(\mu)}$, $Q^T_{(\mu)}Q_{(\mu)} = I$, to find that

(15.194) $$I = I + \epsilon(C_{(\mu)} + C^T_{(\mu)}) + \cdots$$

Equating powers of ϵ we get

(15.195) $$C^T_{(\mu)} = -C_{(\mu)}$$

which is simply the well-known fact that infinitesimal rotations are generated by antisymmetric matrices. We therefore have

(15.196) $$Q^T_{(\mu)} = Q^{-1}_{(\mu)} = I - \epsilon C_{(\mu)} + \cdots$$

The above result will enable us to calculate the derivative of the matrix p, $p_{|\mu}$, at P. Using (15.193) and (15.196), we obtain $p^{(\mu)}$ at $P(\epsilon,\mu)$ as

(15.197) $$p^{(\mu)} = Q^T_{(\mu)}\tilde{p}Q_{(\mu)} = \tilde{p} + \epsilon(\tilde{p}C_{(\mu)} - C_{(\mu)}\tilde{p}) + O(\epsilon^2)$$

The derivative of $p^{(\mu)}$ at P is thus easily obtained by writing

(15.198) $$\frac{p^{(\mu)} - \tilde{p}}{\epsilon} = (\tilde{p}C_{(\mu)} - C_{(\mu)}\tilde{p}) + O(\epsilon)$$

and taking the limit as $\epsilon \to 0$,

(15.199) $$p_{|\mu} = \lim_{\epsilon \to 0} \frac{p^{(\mu)} - \tilde{p}}{\epsilon} = \tilde{p}C_{(\mu)} - C_{(\mu)}\tilde{p}$$

Let us write C_μ in terms of 2×2 matrices as

(15.200) $$C_{(\mu)} = \begin{pmatrix} a_{(\mu)} & b_{(\mu)} \\ c_{(\mu)} & d_{(\mu)} \end{pmatrix}$$

where the antisymmetry of $C_{(\mu)}$ implies that $a^T_{(\mu)} = -a_{(\mu)}$, $d^T_{(\mu)} = -d_{(\mu)}$, and $b^T_{(\mu)} = -c_{(\mu)}$. Then (15.199) takes the simple form

(15.201) $$p_{|\mu} = i\begin{pmatrix} 0 & Jb_{(\mu)} \\ -c_{(\mu)}J & 0 \end{pmatrix}$$

In precisely the same manner we obtain the derivative of q at P:

(15.202) $$q_{|\mu} = i\begin{pmatrix} 0 & -b_{(\mu)}J \\ Jc_{(\mu)} & 0 \end{pmatrix}$$

Later we shall need the derivative of $\Gamma_{\mu\nu} = c^2T_{\mu\nu}$, so we shall obtain it now while it is most convenient; recall that Γ in terms of p and q is given by

(15.203) $$\Gamma = \tfrac{1}{2}r^2(p^2 - q^2)$$

Using (15.194), (15.195), (15.201), and (15.203), we find that at P

(15.204) $$\begin{aligned}\Gamma_{|\mu} &= 2\tilde{\Gamma}(\log r)_{|\mu} + r^2(\tilde{p}p_{|\mu} - \tilde{q}q_{|\mu}) \\ &= (\log r)_{|\mu}r^2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + r^2\begin{pmatrix} 0 & b_{(\mu)} \\ -c_{(\mu)} & 0 \end{pmatrix}\end{aligned}$$

In the results (15.201) to (15.204), only the 2×2 matrices $b_{(\mu)}$ and $c_{(\mu)} = -b^T_{(\mu)}$ occur. Let us write these as

(15.205) $$b_{(\mu)} = \begin{pmatrix} k_\mu & l_\mu \\ m_\mu & n_\mu \end{pmatrix} \qquad c_{(\mu)} = -\begin{pmatrix} k_\mu & m_\mu \\ l_\mu & n_\mu \end{pmatrix}$$

Substituting these into (15.201) to (15.204), we obtain

(15.206a) $$p_{|\mu} = i\begin{pmatrix} 0 & Jb_{(\mu)} \\ -c_{(\mu)}J & 0 \end{pmatrix} = i\left(\begin{array}{cc|cc} 0 & 0 & m_\mu & n_\mu \\ 0 & 0 & -k_\mu & -l_\mu \\ \hline -m_\mu & k_\mu & 0 & 0 \\ -n_\mu & l_\mu & 0 & 0 \end{array}\right)$$

(15.206b) $$q_{|\mu} = i\begin{pmatrix} 0 & -b_{(\mu)}J \\ Jc_{(\mu)} & 0 \end{pmatrix} = i\left(\begin{array}{cc|cc} 0 & 0 & l_\mu & -k_\mu \\ 0 & 0 & n_\mu & -m_\mu \\ \hline -l_\mu & -n_\mu & 0 & 0 \\ k_\mu & m_\mu & 0 & 0 \end{array}\right)$$

(15.206c) $$\Gamma_{|\mu} = (\log r)_{|\mu}r^2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + r^2\left(\begin{array}{cc|cc} 0 & 0 & k_\mu & l_\mu \\ 0 & 0 & m_\mu & n_\mu \\ \hline k_\mu & m_\mu & 0 & 0 \\ l_\mu & n_\mu & 0 & 0 \end{array}\right)$$

Thus we finally arrive at

(15.207a) $$\Pi_\mu = p_{\mu\nu|\nu} = i\begin{pmatrix} m_3 + n_4 \\ -k_3 - l_4 \\ -m_1 + k_2 \\ -n_1 + l_2 \end{pmatrix}$$

(15.207b) $$K_\mu = q_{\mu\nu|\nu} = i\begin{pmatrix} l_3 - k_4 \\ n_3 - m_4 \\ -l_1 - n_2 \\ k_1 + m_2 \end{pmatrix}$$

We thus have achieved our goal of obtaining an explicit form for the vectors K_μ and Π_μ in terms of the elements of the generating matrix C_μ. Substitute now these terms (15.207) into (15.192*a*) to obtain

(15.208) $$\alpha_{|\mu} = \begin{pmatrix} n_3 - m_4 \\ k_4 - l_3 \\ l_2 - n_1 \\ m_1 - k_2 \end{pmatrix}$$

One should note an extraordinary thing at this point: The vector $\alpha_{|\mu}$ is composed of the same components k_μ, l_μ, m_μ, and n_μ which occur in $\Gamma_{|\mu}$. This is a very important observation, and the only problem remaining before we obtain the final Wheeler-Misner condition is to express this correspondence in a covariant manner. With this goal in mind, let us compute the covariant vector

(15.209) $$v_\lambda = \frac{\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma}{\Gamma_{\rho\kappa}\Gamma^{\rho\kappa}}$$

in our special coordinate system at the world-point P. [Recall that, in Eq. (3.25a), we showed that $\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}$ is a tensor.] Computation of the denominator is immediate from (15.172):

$$\tilde{\Gamma}_{\rho\kappa}\tilde{\Gamma}^{\rho\kappa} = \mathrm{Tr}\,(\Gamma^2) = r^4 \tag{15.210}$$

Next observe that the numerator in (15.209) takes the simple form in our special coordinate system

$$\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma = \epsilon_{\lambda\nu\beta\gamma}\Gamma_{\beta\mu|\nu}\tilde{\Gamma}_{\mu\gamma} \tag{15.211}$$

which, from (15.206c), is

$$\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma = \epsilon_{\lambda\nu\beta\gamma}\left[2\tilde{\Gamma}_{\beta\mu}(\log r)_{|\nu} + r^2\begin{pmatrix} 0 & b_{(\mu)} \\ -c_{(\mu)} & 0 \end{pmatrix}\right]\tilde{\Gamma}_{\mu\gamma} \tag{15.212}$$

Note, however, that the Rainich condition (15.164) tells us that $\tilde{\Gamma}_{\beta\mu}\tilde{\Gamma}_{\mu\gamma}$ is a multiple of the identity matrix; thus, since $\epsilon_{\lambda\nu\beta\gamma}$ is antisymmetric in β and γ, the first term of (15.212) obviously vanishes and we are left with

$$\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma = \epsilon_{\lambda\nu\beta\gamma}\begin{pmatrix} 0 & b_{(\mu)} \\ -c_{(\mu)} & 0 \end{pmatrix}\tilde{\Gamma}_{\mu\gamma}r^2 \tag{15.213}$$

Substituting now $\tilde{\Gamma}_{\mu\nu}$ from (15.185) and $b_{(\mu)}$ and $c_{(\mu)}$ from (15.205) into this expression, we obtain

$$\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma = \tfrac{1}{2}r^4\begin{pmatrix} 0 & 0 & -k_\nu & -l_\nu \\ 0 & 0 & -m_\nu & -n_\nu \\ k_\nu & m_\nu & 0 & 0 \\ l_\nu & n_\nu & 0 & 0 \end{pmatrix} \tag{15.214}$$

and hence, by a slightly tedious but elementary calculation,

$$v_\lambda = \frac{\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma}{\Gamma_{\rho\kappa}\Gamma^{\rho\kappa}} = \begin{pmatrix} n_3 - m_4 \\ k_4 - l_3 \\ l_2 - n_1 \\ m_1 - k_2 \end{pmatrix} \tag{15.215}$$

Comparing this with (15.208), we see that we have achieved our goal, for at P in our special coordinate system,

$$\alpha_{|\lambda} = v_\lambda \equiv \frac{\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}\Gamma^{\beta\mu\|\nu}\Gamma_\mu{}^\gamma}{\Gamma_{\rho\kappa}\Gamma^{\rho\kappa}} \tag{15.216}$$

This, however, is written in tensor form and is thus true in general. In view of the simple relation (15.160) between the tensors $\Gamma^{\mu\nu}$ and $T^{\mu\nu}$, we may replace in this identity $\mathbf{\Gamma}$ by $\mathbf{T}$. By use of the Einstein equations (15.2a), this may be put into a purely geometrical form:

$$\alpha_{|\lambda} = v_\lambda = \frac{\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}R^{\beta\mu\|\nu}R_\mu{}^\gamma}{R_{\rho\kappa}R^{\rho\kappa}} \tag{15.217}$$

which is our final result.

The basic equation of Wheeler and Misner follows immediately from (15.217), for the condition that (15.217) be integrable is

$$v_{\lambda|\tau} - v_{\tau|\lambda} = 0 \tag{15.218}$$

This constitutes six additional differential conditions on $R_{\mu\nu}$ in order that it correspond to an electromagnetic field.

In conclusion, let us restate the conditions of Rainich, Wheeler, and Misner from (15.163), (15.165), (15.179), and (15.218):

$$R_{\mu\nu}R^\nu{}_\alpha = \tfrac{1}{4}(R_{\tau\beta}R^{\tau\beta})g_{\mu\alpha} \tag{15.219a}$$

$$R_{00} \leq 0 \qquad \text{(in real coordinates)} \tag{15.219b}$$

$$R^\mu{}_\mu = 0 \tag{15.219c}$$

$$v_{\lambda|\tau} - v_{\tau|\lambda} = 0 \qquad v_\lambda = \frac{\sqrt{-g}\,\epsilon_{\lambda\nu\beta\gamma}R^{\beta\mu\|\nu}R_\mu{}^\gamma}{R_{\rho\kappa}R^{\rho\kappa}} \tag{15.219d}$$

These four sets of conditions are the basis of Wheeler's "already unified" field theory. Note that, since (15.219a) and (15.219d) are quadratic in $R_{\mu\nu}$, only the inequality (15.219b) serves to determine the overall sign of $R_{\mu\nu}$.

Let us now take the following new point of view: Suppose we are given the system (15.219) to begin with and are asked to calculate $g_{\mu\nu}$. In order to simplify the problem we might introduce a fictitious new tensor $F_{\mu\nu}$ which satisfies Maxwell's equations and then define a symmetric tensor $T_{\mu\nu}$ as in (15.2a). Finally, we should set $R_{\mu\nu}$ proportional to $T_{\mu\nu}$

and try to solve the resulting system. All this is possible by virtue of (15.219). [Such a procedure would reproduce system (15.2) by retracing our steps in this section which led to (15.219).] This point of view makes it clear that one might wish to think of $F_{\mu\nu}$ as only a convenient mathematical construct.

The procedure described in the preceding paragraph is reminiscent of a standard artifice in two-dimensional potential theory. If we wish to solve Laplace's equation for the unknown $u(x,y)$,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{15.220}$$

it is often very useful to introduce an auxiliary function $v(x,y)$ by the definition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{15.221}$$

That such a function exists follows from the differential equation (15.220) itself. It also follows that v satisfies Laplace's equation, $\nabla^2 v = 0$. Thus, instead of dealing with one second-order differential equation for u, we have achieved a reduction and splitting into the two first-order differential equations (15.221) at the price of introducing the fictitious unknown function $v(x,y)$. In almost all applications of Laplace's equation the function $v(x,y)$ has a simple and important physical interpretation. For example, in fluid dynamics, where u is the velocity potential of a flow, v will play the role of the stream function.

We may interpret the electromagnetic field in the already unified field theory in a way which is analogous to that of $v(x,y)$ in potential theory. Consider (15.219) as the basic system of equations for the $R_{\mu\nu}$ tensor. This entirely geometrical system is, unfortunately, very nonlinear and difficult to handle mathematically. In order to solve it, introduce the $F_{\mu\nu}$ tensor as previously indicated. This splits the nonlinear system (15.219) into a different set of equations (15.2) which are linear in $R_{\mu\nu}$ and not so difficult to solve. In Wheeler's viewpoint the simplification achieved by this reduction is so tremendous that for more than a century physicists have ascribed a physical reality to $F_{\mu\nu}$ and have assumed the existence of an electromagnetic field independent of the metric structure of space-time.

There remains now one further problem in this approach to a unified field theory: How do we describe and explain charged matter, i.e., the sinks and sources of the field? The basic relations in (15.219) are valid only at places where the field is regular and no charge exists. Thus it

would appear necessary to admit singularities in the field as in Sec. 15.1. There is, however, one possibility of geometrizing away even the singularities. As mentioned above, the relations (15.219) lead back to Maxwell's equations, and these in turn lead to the concept of lines of force which can end only at singularities of the field. We need these singularities, therefore, to serve as sinks and sources for lines of force; more mathematically, we need singularities only in order to circumvent the uniqueness theorems on the solutions of partial-differential-equation systems, and thereby exclude the trivial solution of an empty and static world. Wheeler accomplishes this end without introducing singularities by endowing the world with an appropriate novel topology. Suppose, for example, that in first approximation the world is a sphere in four-space. One may deform the sphere by adding a handle between world-points P_1 and P_2 as shown in Fig. 15.2. Such a deformation gives the world the topological aspect of a four-dimensional beer stein. Lines of force could now be drawn on the original sphere which would disappear at P_1 by entering the handle and reissue at P_2. Then P_1 would be a sink and P_2 a source, but no singularity would occur. In similar fashion,

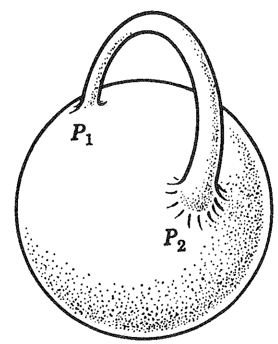

Fig. 13.2

many sources and sinks for the gravitational and electromagnetic fields could be created by an appropriate adjustment of the topology of the world.

Since the topology of a surface is part of its geometry, the field equations would determine the development of geometry and topology in time. That is, we could calculate how the sources and sinks move about in time, and therefore the motion of the particles represented by these sources and sinks would be a consequence of the field equations.

This is an imposing program for the geometrization of classical physics.

It is clear that the actual carrying out of this program will run into great mathematical difficulties, and one cannot yet say that the basic concepts have added anything to our understanding of the physical world. Such ideas may be considered, however, as good examples of the many possibilities hidden in the mathematical structure and geometric concepts which lie at the base of general relativity theory.

Exercises

15.1 An interesting interpretation of the Reissner-Nordstrom metric can be obtained. To see this consider the energy density of the electric field surrounding a point particle. Obtain the total mass-equivalent energy inside a sphere of radius r using the Einstein relation $E = mc^2$, and use Gauss' law to show that its gravitational potential falls off like r^{-2}. If this extra potential is added to the usual point-mass potential in the equation (4.142) for an approximate g_{00}, the result is identical with g_{00} in (15.21).

15.2 Show that for $(\epsilon/M)^2/\kappa < 1/4\pi$ the Reissner-Nordstrom metric has a spherical null surface or one-way membrane, as discussed in Sec. 7.8, while for $(\epsilon/M)^2/\kappa > 1/4\pi$ it does not. What of $(\epsilon/M)^2/\kappa = 1/4\pi$?

15.3 Calculate the Riemann tensor $R^\alpha{}_{\beta\gamma\eta}$ for the Reissner-Nordstrom metric and show that it is singular only at $r = 0$.

15.4 Study the geodesics in the Reissner-Nordstrom metric, in particular the radial null geodesics. How do the geodesics behave in the region of the null surfaces discussed above?

15.5 If $(\epsilon/M)^2/\kappa > 1/4\pi$, the singularity of the Reissner-Nordstrom metric at $r = 0$ is termed *naked* since it is not surrounded by a null surface. Show that light rays, or null geodesics, can pass from the neighborhood of this singularity to large values of r, for example, $r > 2m$, in a finite amount of coordinate time. Hence the region of the singularity is visible to an exterior observer, unlike the situation in the Schwarzschild case.

15.6 Show that the Reissner-Nordstrom metric can be put into the form $\eta_{\mu\nu} - 2ml_\mu l_\nu$ with $l^\mu l_\mu = 0$, as discussed in Chap. 7, by a transformation of the time coordinate (see Exercise 7.11).

15.7 Show that the self-dual Riemann tensor and the self-dual Weyl tensor are equal if $R = 0$ and it is not necessary that $R_{\mu\nu} = 0$. Thus the Petrov classification in the presence of the electromagnetic field can be made with the Riemann tensor, which is simpler than the Weyl tensor (see Exercise 10.8).

15.8 What is the Petrov type of a space-time with a Reissner-Nordstrom metric?

Problems

15.1 A solution of the Einstein-Maxwell equations has been found by Newman and collaborators (1965) that is the generalization of the Reissner-Nordstrom metric in the same sense that the Kerr metric is the generalization of the Schwarzschild metric. It represents the field of a spinning charged body. It is

$$ds^2 = \left(1 - \frac{2mr - e^2}{r^2 + a^2\cos^2\theta}\right) c^2\,dt^2 - \frac{r^2 + a^2\cos^2\theta}{\Delta}\,dr^2 - (r^2 + a^2\cos^2\theta)\,d\theta^2 - \left(r^2 + a^2 + \frac{(2mr - e^2)a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta\,d\varphi^2 - \frac{2a(2mr - e^2)}{r^2 + a^2\cos^2\theta}\sin^2\theta\,d\varphi\,c\,dt$$

where $\Delta \equiv r^2 - 2mr + e^2 + a^2$. Show that this is indeed a solution of the Einstein-Maxwell equations.

15.2 Study the singularities and the existence of null surfaces in the Kerr-Newman metric introduced above.

15.3 What is the asymptotic magnetic field of the Kerr-Newman metric for large r? What is the effective magnetic moment corresponding to this field? Show that the ratio of magnetic moment to angular momentum is ϵ/M; this is twice the so-called "normal" value that one obtains classically for any distribution of material with a constant ratio of charge to mass density but is the same as the ratio obtained for an electron in Dirac's relativistic quantum theory.

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Index

Index